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Finitely Presented Semigroups and Associative Algebras*

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1. INTRODUCTION

1.1. In his paper [23] Lewin proved that there exist continuously many 2-generator associative algebras of a very special kind. Thus finitely generated associative algebras which are not finitely presented abound. The object of this paper is to delve a little deeper into the nature of finitely presented associative algebras (as well as finitely presented semigroups). The direction that I have taken is largely motivated by group theory.

1.2. Let me recall first some definitions. Suppose A is an associative algebra over a field $\mathbf{k}$. If $a, b \in A$ then we put

$$[a, b] = ab - ba,$$

the so-called *commutator* of a and b. If X and Y are nonempty subsets of A, then we denote by $[X, Y]$ the ideal of A generated by the commutators $[x, y]$ $(x \in X, y \in Y)$. As usual $[A, A]$ is called the *commutator* or *derived ideal* of A; we denote $[A, A]$ also by A'. If B is an ideal of A and n a positive integer, then the subalgebra B^n of A generated by the n-fold products of elements of B is again an ideal of A. A is termed *nilpotent* if $A^{n+1} = 0$ with the least such n the *class* of A.

An associative algebra A can be turned into a Lie algebra A° by using the same vector space structure as A and introducing a second binary operation $\circ$ in A defined by

$$a \circ b = [a, b].$$

We say A is *Lie nilpotent* if A° is a nilpotent Lie algebra.

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If **U**, **V** are classes of (associative) algebras, we term A a **U**-by-**V** algebra if there is a short exact sequence

$$0 \to X \to A \to Y \to 0$$

with $X \in \mathbf{U}$ and $Y \in \mathbf{V}$. In particular A is termed *bicommutative* (or *metabelian* according to K. W. Gruenberg [15]) if it is a **C**-by-**C** algebra where **C** denotes the class of all commutative algebras.

1.3. For completeness we begin with a discussion of the number of finitely generated algebras, *over a countable field*. There are, of course, a countable number of finitely presented algebras. By simply going over to group algebras one can concoct a host of finitely generated infinitely related algebras (see, e.g., Theorem A of Baumslag [6]). Here we restrict our attention to solvable algebras in the sense of Jennings [18] (see Section 2.1 for the relevant definition). In fact, there are continuously many 2-generator algebras A satisfying $(A')^3 = 0$ (Lewin [23]). One can bypass Lewin by copying an argument of Hall [16] for groups which yields the slightly stronger

THEOREM 1. *There are continuously many 2-generator algebras A satisfying $(A')^2 A = 0 = A(A')^2$.*

Theorem 1 is best possible in the sense that we prove, in Section 2.3, the following generalization of the Hilbert Basis Theorem.

THEOREM 2. *Let A be a finitely generated algebra, B a commutative ideal of A. If A/B is finitely presented and Noetherian, then A is Noetherian.*

One consequence of Theorem 2 is

COROLLARY 2.1. *A finitely generated commutative-by-Lie nilpotent algebra is Noetherian.*

It follows easily that

COROLLARY 2.2. *There are countably many finitely generated commutative-by-Lie nilpotent algebras.*

Note that Corollary 2.2 applies in particular to bicommutative algebras. Such algebras were first studied in detail by Gruenberg [15] in an unpublished essay in 1951.

1.4. We turn next, in Section 3, to an attempt to discern some of the finitely generated algebras which are not finitely presented.

To put our results into perspective, suppose A is an augmented $\mathbf{k}$-algebra, F a free associative algebra with 1, and that $\mathbf{A}$ is the augmentation ideal of F. If

$$A \cong F/R$$

is a presentation of A, then

$$(\mathbf{A}^2 \cap R)/(R\mathbf{A} + \mathbf{A}R) \cong H_2(A, \mathbf{k}),$$

the second homology group of A with coefficients in the trivial A-module $\mathbf{k}$ (Frohlich [14]). If A is finitely presented then it follows that $H_2(A, \mathbf{k})$ is finite dimensional. If $A = \mathbf{k}G$ is the group algebra of the group G over $\mathbf{k}$, then A is an augmented $\mathbf{k}$-algebra, and so $H_2(A, \mathbf{k}) = H_2(G, \mathbf{k})$ is simply the second homology group of G with coefficients in the trivial G-module $\mathbf{k}$. Since there exist finitely generated but infinitely related groups G satisfying $H_2(G, \mathbf{k}) = 0$ (see, e.g., [3]) it follows from [6] that there exist finitely generated augmented $\mathbf{k}$-algebras A with $H_2(A, \mathbf{k}) = 0$, which are not finitely presented.

Our concern here is with associative algebras where no assumptions are made as to augmentedness or to the existence of a 1. Suppose now that F denotes a free associative $\mathbf{k}$-algebra without 1, A an arbitrary $\mathbf{k}$-algebra. Then if

$$A \cong F/R$$

is a presentation of A,

$$(F^2 \cap R)/(FR + RF)$$

is again a homology group of A and hence an invariant (see Frohlich [14], Knopfmacher [20]). We denote this invariant simply by $m(A)$ and observe again that if A is finitely presented then $m(A)$ is finite dimensional. As in group theory, we term $m(A)$ the *multiplicator* of A.

Now suppose we write

$$A = \langle\!\langle x_1, ..., x_p; r_1, ..., r_q \rangle\!\rangle$$

to express the fact that the algebra A can be generated by its elements $x_1, ..., x_p$ and defined in terms of them by all relations of the form

$$[a_1[b, c]\, a_2, a_3[b_1, c_1]\, a_4] = 0 \qquad (a_1, a_2, a_3, a_4, b, c, b_1, c_1 \in A)$$

(i.e., those relations which ensure that A is bicommutative) together with the relations $r_1 = r_2 = \cdots = r_q = 0$. Then (see Section 3.1)

THEOREM 3. *Let*

$$A = \langle\!\langle x_1, ..., x_p; r_1, ..., r_q \rangle\!\rangle \qquad (p, q \textit{ finite}).$$

If $p - q \geqslant 2$ and $r_1, ..., r_q$ are all commutator words, i.e., they lie in the derived *ideal of the free algebra on* $x_1, \ldots, x_p$, *then* $m(A)$ *is not finite dimensional* (*and hence* A *is not finitely presented*).

Now an algebra of the form F/R, where R is the ideal of F generated by $(F')'$ is termed *free bicommutative* (cf. the remarks below in Section 1.5). It therefore follows immediately from Theorem 3 that

COROLLARY 3.1. *A noncommutative free bicommutative algebra has an infinite dimensional multiplicator.*

1.5. There is another application of multiplicators to the study of finitely presented algebras involving *parafree algebras*. This notion is a relative one involving a *variety* $\mathbf{V}$ of algebras. By definition $\mathbf{V}$ is a family of algebras closed under epimorphic images, subalgebras, and products (i.e., unrestricted direct products). We take for granted here the usual facts about varieties as well as the terminology such as "free algebra in a variety $\mathbf{V}$" (see, e.g., Cohn [12]). Recall next that an algebra A is *residually nilpotent* if $\bigcap_{n=1}^{\infty} A^n = 0$.

Now suppose $\mathbf{V}$ is a variety of algebras in which the free algebras are residually nilpotent. We term an algebra A $\mathbf{V}$-*parafree* or simply *parafree*, if $\mathbf{V}$ is understood to be the relevant variety, if the following three conditions hold:

(1) A belongs to $\mathbf{V}$;

(2) A is residually nilpotent;

(3) there exists a free algebra F in $\mathbf{V}$ such that

$$A/A^n \cong F/F^n \qquad (n = 1, 2, \ldots).$$

If F is of rank q, then A is termed $\mathbf{V}$-parafree of rank q.

It is easy to establish the existence of a wide assortment of parafree algebras by simply tracing through the corresponding arguments for groups. We give here two samples. The first of these is (cf. [2])

THEOREM 4. *Let* $\mathbf{V}$ *be a variety of associative algebras in which the free algebras are residually nilpotent and the algebra of rank* 1 *is not nilpotent. Then for every positive integer* q *there exists a colimit of free algebras in* $\mathbf{V}$ *which is parafree of rank* q *but not free.*

The second family of parafree algebras is probably well known. We establish their existence here by translating the corresponding argument for groups due to Bousfield and Kan [9] (see Baumslag and Stammbach [8]) yielding

THEOREM 5. *Let* $\mathbf{V}$ *be a variety of algebras in which the free algebras are residually nilpotent. If* F *is a finitely generated free algebra in* $\mathbf{V}$ *then its pronilpotent completion* $\hat{F}$ *is parafree in* $\mathbf{V}$.

It is easy to prove that if A is an absolutely free algebra (i.e., free in the variety of all algebras) then $m(A) = 0$. I do not know whether this is also true for the corresponding absolutely parafree algebras, but it seems unlikely. Our main objective involves the parafree algebras in the variety of all bicommutative algebras. Indeed, we prove the following generalization of Corollary 3.1.

THEOREM 6. *Let* $\mathbf{M}$ *be the variety of all bicommutative algebras. Then the multiplicator of a parafree algebra in* $\mathbf{M}$ *of rank at least two is not finite dimensional.*

1.6. Every finitely generated metabelian group czn be embedded in a finitely presented metabelian group (Baumslag [4]). The principal purpose of this paper is to prove some theorems of a similar nature for associative algebras. Since associative algebras are structurally richer than groups, it is not surprising that the associative algebra versions of these theorems (see Section 5) encompass somewhat more than the corresponding known theorems for groups (see [4, 17, and 27]).

All of our embedding theorems for associative algebras stem from a simple criterion, Theorem 7 (below), for an associative algebra to be finitely presented. To explain, we need to introduce a definition. Let then T be an associative algebra over (as always) a commutative field. Now let M be a (T, T)-bimodule. We term M *ample* if for each $m \in M$, $t \in T$, there exist t', $t'' \in T$ such that

$$mt = t'm, \qquad tm = mt''.$$

The following theorem then bolds.

THEOREM 7. *Let*

$$0 \to N \to A \to T \to 0$$

be a short exact sequence of associative algebras, where N *is nilpotent and all of the quotients of* T *are finitely presented. If* N^i/N^{i+1}*, viewed as a* (T, T)*-bimodule, is finitely generated and ample for every* i*, then* A *is finitely presented.*

On appealing to a theorem of Lewin [24], one can use Theorem 7 to deduce

THEOREM 8. *Let* A *be a finitely generated associative algebra with* 1 *containing an ideal* N *with* $N^2 = 0$*. If every quotient of* A/N *is finitely presented then* A *can be embedded in a finitely presented algebra* A^* *which closely resembles* A*; specifically* A^* *contains an ideal* N^* *such that*

$$\text{(i)}\quad N^{*2} = 0 \qquad \textit{and} \qquad \text{(ii)}\quad A^*/N^* = A/N \otimes A/N.$$

Theorem 8 has a number of interesting consequences.

COROLLARY 8.1. *Let A be a finitely generated associative algebra which is the middle of a short exact sequence*

$$0 \to N \to A \to T \to 0$$

in which $N^2 = 0$ and T is Lie-nilpotent. Then A can be embedded in a finitely presented algebra A^* which is also the middle of a short exact sequence of exactly the same kind.

Among the Lie-nilpotent algebras are the commutative algebras. So a special case of Corollary 8.1 is the following:

COROLLARY 8.2. *Suppose the finitely generated associative algebra A contains an ideal N such that $N^2 = 0$ and A/N is commutative. Then A can be embedded in a finitely presented algebra A^* containing an ideal N^* such that $N^{*2} = 0$ and A^*/N^* is commutative.*

Corollary 8.2 may be regarded as an analog of the embedding theorem for finitely generated metabelian groups cited above.

We turn now to a slightly different runoff from Theorem 7, viz.,

THEOREM 9. *Let A be a finitely generated algebra of lower triangular matrices with coefficients in a commutative algebra B. Then A can be embedded in a finitely presented algebra of lower triangular matrices with coefficients in B.*

The analog of Theorem C has recently been obtained for groups by Thomson [27].

Our last embedding theorem for associative algebras is

THEOREM 10. *Let F be a finitely generated free associative algebra with 1, and let R be an ideal of F. If F/R is finitely presented, then F/R^n can be embedded in a finitely presented algebra A containing an ideal S such that*

$$\text{(i)} \quad S^n = 0 \qquad \textit{and} \qquad \text{(ii)} \quad A/S = F/R \otimes F/R.$$

Unlike Theorem 8 no analog of Theorem 10 has as yet been proved for groups.

1.7. We turn next to three theorems concerned with the finite presentability of certain semigroups.

First we record the simple

THEOREM 11. *Let S be a semigroup, $\mathbf{k}$ a field. Then the semigroup algebra $\mathbf{k}S$ is finitely presented (as an associattive $\mathbf{k}$-algebra) if and only if S is finitely presented.*

Theorem 11 is the anaolog of the corresponding theorem for groups proved in

[6], and its proof can be constructed along similar lines. The ground field $\mathbf{k}$ plays no real role here and can be replaced by an arbitrary ring. It should also be pointed out that Theorem 11 is a generalization of a theorem of Rabin [26].

Our second theorem involves the notion of parafreeness discussed in Section 1.5. In order to explain we recall that an element 0 in a semigroup S is termed a *zero* if

$$0s = 0 = s0 \qquad (s \in S).$$

Of course this element 0 is uniquely defined by this condition. For each such semigroup S and every positive integer n, we define

$$\mathbf{c}_n(S) = \{(a, b) \in S \times S \mid a = b \text{ or } a = a_1 \cdots a_n\,,\ b = b_1 \cdots b_n\ (a_1\,,\ldots, a_n\,, b_1\,,\ldots, b_n \in S)\}.$$

It is easy to see that $\mathbf{c}_n(S)$ is a congruence in S and the quotient semigroup $S/\mathbf{c}_n(S)$ is *nilpotent*; i.e., the product of any m elements of $S/\mathbf{c}_n(S)$ is zero for some $m \geqslant 1$, indeed here we can take $m = n$. This semigroup $S/\mathbf{c}_n(S)$ is called the nth *lower central quotient of* S. We term S *residually nilpotent* if

$$\bigcap_{n=1}^{\infty} \mathbf{c}_n(S) = \{(s, s) \mid s \in S\}.$$

Note that this means that if $s \neq t$ then

$$(s, t) \notin \mathbf{c}_n(S) \qquad \text{for some } n \geqslant 1.$$

Finally, we term a semigroup S with zero *parafree* if

(i) S is residually nilpotent;

(ii) there exists a free semigroup F with zero such that

$$S/\mathbf{c}_n(S) \cong F/\mathbf{c}_n(F)$$

for every $n = 1, 2,\ldots.$

As already noted parafree algebras which are not free exist in abundance. By way of contrast, however, we shall here prove

THEOREM 12. *Every parafree semigroup is a free semigroup with zero.*

Our final result about semigroups is motivated by our earlier embedding theorems for algebras. To this end let $\mathscr{X}$ be the class of those semigroups S satisfying the following conditions:

(1) S is finitely generated;

(2) $S = A \cup T$, where A and T are disjoint commutative subsemigroups of S;

(3) A is an ideal of S, i.e., if $a \in A$, $s \in S$ then $as \in A$ and $sa \in A$.

The semigroups in $\mathscr{X}$ may be likened to finitely generated bicommutative algebras. Now although the semigroups in $\mathscr{X}$ are not all finitely presented it turns out that

THEOREM 13. *Every semigroup in $\mathscr{X}$ can be embedded in a finitely presented semigroup in $\mathscr{X}$.*

2. SOLVABLE ASSOCIATIVE ALGEBRAS

2.1. *Preliminaries*

Throughout A denotes an associative algebra over a commutative field $\mathbf{k}$; we emphasize that no assumption is made about the existence of a 1 in A. The *Lie lower central series*

$$A = A_1 \geqslant A_2 \geqslant \cdots$$

of A is defined by

$$A_{i+1} = [A_i, A] \qquad (i = 1, 2, \ldots);$$

we recall from Section 1.2 that A_{i+1} is the *ideal of A* generated by the commutators $[x, y]$ $(x \in A_i, y \in A)$. It turns out (Jennings [19]) that if A is finitely generated and Lie nilpotent, then the Lie lower central series of A terminates in 0 in a finite number of steps. Similarly the derived series

$$A = A^{(0)} \geqslant A^{(1)} \geqslant \cdots$$

of A is defined by

$$A^{(i+1)} = [A^{(i)}, A^{(i)}] \qquad (i = 1, 2, \ldots).$$

Thus $A^{(i+1)}$ is the *ideal of A* generated by the commutators $[x, y]$ $(x, y \in A^{(i)})$. A is termed *solvable* if $A^{(n)} = 0$ for some n.

We shall need the following:

LEMMA 1. *Let*

$$0 \to B \to A \to C \to 0$$

be a short exact sequence of algebras. Then

(i) *if B and C are finitely presented, so is A;*

(ii) *if C is Noetherian and every ascending chain of ideals of A contained in B is finite, then A is Noetherian.*

The proof of Lemma 1 is straightforward and so is omitted (cf. Lemma 1 of Hall [16]).

Now if A is a finitely generated algebra then A_i/A_{i+1} is a finitely generated commutative algebra for every i (Gruenberg [15])—this can be proved directly by a simple "straightening process" and induction. Notice also that a finitely generated commutative algebra is finitely presented, by the Hilbert basis theorem. These remarks, together with Lemma 1, easily yield

LEMMA 2. *Let A be a finitely generated Lie nilpotent algebra. Then the following hold:*

(i) *every quotient of A is finitely generated and Lie nilpotent;*

(ii) *A and also its quotients are finitely presented;*

(iii) *A is Noetherian.*

We write $B \trianglelefteq A$ to express the fact that B is an ideal of A. The following lemma is an immediate consequence of the hypothesis.

LEMMA 3. *Suppose $B \trianglelefteq A$, A is finitely generated, and that A/B is finitely presented. Then B is a finitely generated ideal, i.e., the smallest ideal of A containing a suitably chosen finite set of elements of B.*

2.2. *Counting Finitely Generated Solvable Algebras*

Lewin [23] has proved that there are continuously many 2-generator algebras A satisfying $(A')^3 = 0$. As Lewin points out, this condition cannot be relaxed to $(A')^2 = 0$. However we have

THEOREM 1. *There are continuously many 2-generator algebras A satisfying $(A')^2 A = 0 = A(A')^2$.*

Proof. Let $\mathbf{k}[x]$ be the polynomial algebra in x over the given ground field $\mathbf{k}$. Put

$$s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad a = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and let A be the algebra generated by s and a. A direct calculation shows

$$(A')^2 A = 0 = A(A')^2.$$

Moreover $(A')^2$ turns out to be infinite dimensional. Since every subspace of $(A')^2$ is an ideal of A, it follows that A has continuously many ideals and hence continuously many nonisomorphic quotients, as desired.

2.3. *Counting Finitely Generated Solvable Algebras (Continued)*

Our next result centers around the following:

LEMMA 4. *Let $B \trianglelefteq A$, and suppose that B is commutative. If $b_1, b_2 \in B$ and $t_1, ..., t_n \in A$, then for all $r, s, k, l \in \{1, ..., n\}$ $(r \leqslant s,\ k \leqslant l)$, all permutations $i_1 i_2 \cdots i_n$ of $1, 2 \cdots n$ and $j_1 j_2$ of $1\ 2$, the following holds*

$$t_1 t_2 \cdots t_r b_1 t_{r+1} \cdots t_s b_2 t_{s+1} \cdots t_n = t_{i_1} t_{i_2} \cdots t_{i_k} b_{j_1} t_{i_{k+1}} \cdots t_{i_l} b_{j_2} t_{i_{l+1}} \cdots t_{i_n}. \qquad (1)$$

Proof. We proceed by induction on n.

If $n = 0$ the result follows from the commutativity of B. Suppose $n > 0$. Consider the left-hand side of (1):

$$\begin{aligned} t_1 t_2 \cdots t_r b_1 t_{r+1} \cdots t_s b_2 t_{s+1} \cdots t_n &= t_1 t_2 \cdots t_r t_{r+1} \cdots t_s b_2 t_{s+1} \cdots t_n b_1, \\ &= t_1 t_2 \cdots t_r t_{r+1} \cdots t_s t_{s+1} \cdots t_n b_1 b_2, \\ &= t_1 t_2 \cdots t_n b_{j_1} b_{j_2}. \end{aligned}$$

Similarly we find that the right-hand side of (2) can be expressed in the form $t_{i_1} t_{i_2} \cdots t_{i_n} b_{j_1} b_{j_2}$. If $i_1 = 1$, then the inductive hypothesis applied to $t_{i_2} \cdots t_{i_n} b_{j_1} b_{j_n}$ yields the desired equality. If $i_1 \neq 1$, then we proceed as follows:

$$\begin{aligned} t_{i_1} t_{i_2} \cdots t_{i_n} b_{j_1} b_{j_2} &= t_{i_1} t_{q_2} \cdots t_{q_n} b_{j_1} b_{j_2} \qquad (1 = q_2 < q_3 < \cdots < q_n) \text{ inductively}, \\ &= t_{i_1} b_{j_2} t_{q_2} \cdots t_{q_n} b_{j_1}, \\ &= t_{q_2} \cdots t_{q_n} b_{j_1} t_{i_1} b_{j_2}, \\ &= t_{q_2} \cdots t_{q_n} t_{i_1} b_{j_2} b_{j_1}, \\ &= t_1 t_2 \cdots t_n b_{j_1} b_{j_2} \qquad \text{inductively}. \end{aligned}$$

This completes the proof.

COROLLARY 4.1. *If B is a commutative ideal of A contained in A', then $B^3 = 0$.*

Proof. Every element of B^3 can be written as a k-linear combination of products p of the form

$$p = b_1 t_1[x_1, y_1]\, t_1' t_2[x_2, y_2]\, t_2' \cdots t_n[x_n, y_n]\, t_n' b_2 (b_1, b_2 \in B,\ t_i, t_i', x_i, y_i \in A).$$

By Lemma 4,

$$p = t_1 t_1' b_1[x_1, y_1]\, b_2 t_2[x_2, y_2]\, t_2' \cdots t_n[x_n, y_n]\, t_n' .$$

Since $n \geqslant 1$, the factor $[x_1, y_1]$ actually appears. Now

$$\begin{aligned} b_1[x_1, y_1]\, b_2 &= b_1(x_1 y_1 - y_1 x_1)\, b_2, \\ &= b_1 x_1 y_1 b_2 - b_1 y_1 x_1 b_2, \\ &= x_1 y_1 b_1 b_2 - x_1 y_1 b_1 b_2 \qquad \text{(by Lemma 4)}, \\ &= 0. \end{aligned}$$

So $p = 0$.

COROLLARY 4.2. *If A is bicommutative, $(A')^3 = 0$.*

Corollary 4.2 is due to Gruenberg [15]. I discovered it independently some 23 years later (see also Cohn [13, p. 33]). It is not hard to show that the sharper conclusion $(A')^2 = 0$ is not true. However, by way of contrast, one has

COROLLARY 4.3. *Let A be a split extension of B by T:*

$$A = B \oplus T, \qquad (B \trianglelefteq A,\ T \text{ a subalgebra of } A).$$

If both B and T are commutative, then

$$(A')^2 = 0.$$

Proof. If $b_1, b_2 \in B$, $t \in T$ it follows on appealing to Lemma 4 as needed that

$$[b_1, t]\, b_2 = (b_1 t - t b_1)\, b_2 = b_1 t b_2 - t b_1 b_2 = t b_1 b_2 - t b_1 b_2 = 0.$$

Hence, if now $t_1, t_2 \in T$, $a_1, a_2, a_3, a_4 \in A$,

$$a_1[b_1, t_1]\, a_2 a_3 [b_2, t_2]\, a_4 = a_1 a_2 a_3 a_4 [b_1, t_1]\, [b_2, t_2] = 0$$

since $[b_2, t_2] \in B$. It follows that $(A')^2 = 0$ as desired.

We come now to the main application of Lemma 4.

THEOREM 2. *Let A be a finitely generated algebra, B a commutative ideal of A. If A/B is finitely presented and Noetherian, then A is Noetherian.*

Proof. Put

$$C = B \cap A'.$$

Then $C \trianglelefteq A$. Note that

$$B/C \simeq (B + A')/A' \leq A/A'. \tag{2}$$

So B/C is an ideal of the finitely generated commutative **k**-algebra A/A'. Hence, B/C is a finitely generated ideal of A/A' by the Hilbert basis theorem. Consider now the short exact-sequence

$$0 \to B/C \to A/C \to A/B \to 0.$$

We claim that A/C is Noetherian and finitely presented. The first claim follows from Lemma 1(ii) and the Hilbert basis theorem. The proof that A/C is finitely presented is an elaboration of the argument needed to prove Lemma 1(i), keeping in mind that B/C is a finitely generated ideal of A/C, where the action of A/C on B/C is obtained from isomorphism (2). It follows that it suffices to restrict attention to the case where B is contained in A'.

Since A/B is finitely presented B is a finitely generated ideal of A, by Lemma 3. Furthermore, $B^3 = 0$ by Corollary 4.1. It follows that we may view B/B^2 as a finitely generated (A/B)-bimodule. Since A/B is Noetherian, so is this (A/B)-bimodule B/B^2. This ensures that A/B^2 is Noetherian, on appealing to Lemma 1(ii). Now it follows from Lemma 4 and the fact that B/B^2 is a finitely generated (A/B)-bimodule, that B^2 itself may be viewed as a finitely generated A/B-module. Again the hypothesis that A/B is Noetherian yields the necessary condition to be able to apply Lemma 1(ii). Thus A is Noetherian, as claimed.

Since a finitely generated Lie-nilpotent algebra is finitely presented and Noetherian (Lemma 2) it follows immediately from Theorem 2 that

COROLLARY 2.1. *A finitely generated commutative-by-Lie nilpotent algebra is Noetherian.*

3. Discerning Finitely Presented Algebras

3.1. *Bicommutative Algebras Defined by Commutator Words*

Multiplicators of finitely generated bicommutative algebras are easier to handle than other algebras because of the following:

LEMMA 5. *A finitely generated bicommutative algebra A has a finite dimensional multiplicator if and only if every quotient of A has a finite dimensional multiplicator.*

The proof of Lemma 5 is analogous to the proof of the corresponding result for finitely generated metabelian groups (Baumslag [7]). It suffices to note that if $R \trianglelefteq A$, a finitely generated bicommutative algebra, then R is finitely generated as an ideal, by Corollary 2.1. So if $RA = 0 = AR$ it follows that R is finite dimensional, the crucial observation needed for Lemma 5.

We shall need another analog of an easy-to-prove lemma for groups, viz.,

LEMMA 6. *A finitely generated algebra P has a finite dimensional multiplicator if and only if for every short exact sequence*

$$0 \to I \to H \to P \to 0,$$

with (i) *H finitely generated,* (ii) *I an ideal of H satisfying $IH = 0 = HI$, the ideal I is finite dimensional.*

The proof is easy and is omitted.

We are now in a position to prove

THEOREM 3. *Let*

$$A = \langle\langle x_1, \ldots, x_p; r_1, \ldots, r_q \rangle\rangle \qquad (p, q < \infty).$$

If $p - q \geqslant 2$ and $r_1, \ldots, r_q$ are all commutator words then $m(A)$ is not finite dimensional, and hence A is not finitely presented.

Proof. It suffices, by Lemma 5, to prove that there exists a quotient P of A with $m(P)$ infinite dimensional. To this end, let B be the ideal of A generated by $x_2, \ldots, x_p$. Then B^2 is again an ideal of A. Put

$$\bar{A} = A/B^2, \qquad \bar{B} = B/B^2, \qquad \bar{x}_i = x_i + B^2 \qquad (i = 1, \ldots, p).$$

Furthermore, let $\bar{T}$ denote the subalgebra of $\bar{A}$ generated by $\bar{x}_1$. Clearly $\bar{T}$ is the $\mathbf{k}$-subalgebra of the $\mathbf{k}$-algebra $T = \mathbf{k}[\bar{x}_1]$ of all polynomials in $\bar{x}_1$ which consists of polynomials with constant term zero.

Since $\bar{A} = \bar{B} \oplus \bar{T}$ it follows that $\bar{A}$ is the semidirect product of $\bar{B}$ and $\bar{T}$. $\bar{T}$ acts on $\bar{B}$ by left and right multiplication, and this action can be extended to T. This turns $\bar{B}$ into a (T, T)-bimodule and thence in the usual way into a left $(T \otimes_{\mathbf{k}} T^{\mathrm{op}})$-module. Note that $T \otimes_{\mathbf{k}} T^{\mathrm{op}}$ is a polynomial algebra over $\mathbf{k}$ in two (commuting variables), and hence is an integral domain. Let $\mathbf{l}$ be the quotient field of $T \otimes_{\mathbf{k}} T^{\mathrm{op}}$.

Now it follows from the very definition of A that the $(T \otimes_{\mathbf{k}} T^{\mathrm{op}})$-module $\bar{B}$ can be presented on $p - 1$ generators subject to q relations. This implies that $\mathbf{l} \otimes_{\mathbf{k}} \bar{B}$ can be viewed as a quotient of a $(p - 1)$-dimensional vector space by a q-dimensional subspace. Since $p - 1 > q$ this ensures that $\mathbf{l} \otimes_{\mathbf{k}} \bar{B}$ is of positive dimension and therefore has a 1-dimensional quotient space V.

Since $\mathbf{l}$ is the quotient field of $T \otimes_{\mathbf{k}} T^{\mathrm{op}}$, V can be tthought of as a left $T \otimes_{\mathbf{k}} T^{\mathrm{op}}$-module and therefore as a (T, T)-bimodule. If we now form the semidirect product

$$S = V \oplus T$$

of V and T it follows readily that there is a homomorphism φ of $\bar{A}$ into S which injects $\bar{T}$ into T and maps $\bar{B}$ nontrivially onto the (T, T)-submodule W of V. Consequently, putting $P = A\varphi$, we find P is the semidirect product of W and $\bar{T}$.

The structure of V ensures that if $a \in W$, $a \neq 0$, then the $(\bar{T}, \bar{T})$-submodule of W generated by a is free, i.e., the elements

$$\bar{x}_1{}^i \, a \, \bar{x}_1{}^j \qquad (i = 0, 1, \ldots, j = 0, 1, \ldots)$$

are linearly independent over $\mathbf{k}$. This observation is used to prove that $m(P)$ is infinite dimensional and thence, as pointed out at the outset, that $m(A)$ is infinite dimensional, as required. In other words, we are left only with the verification of

LEMMA 7. *$m(P)$ is infinite dimensional.*

Proof. We may identify the vector space V over the field $\mathbf{l}$ with $\mathbf{l}$ itself, viewed of course as a vector space over $\mathbf{l}$ in the usual way. Keeping this in mind consider the $\mathbf{k}$-algebra Q generated by the (3×3)-matrices

$$\begin{pmatrix} 0 & w & 0 \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \quad (w \in W), \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{x}_1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

viewed as elements in the $\mathbf{k}$-algebra of all (3×3)-matrices with coefficients in $\mathbf{l}$. Let R be the subalgebra consisting of those matrices in Q of the form

$$\begin{pmatrix} 0 & 0 & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (r \in \mathbf{l}).$$

Then R is an ideal of Q: indeed

$$RQ = 0 = QR.$$

Observe now that if a is a nonzero element of W and i, j are any nonnegative integers then

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{x}_1{}^i & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{x}_1{}^i a \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{x}_1{}^j & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a\bar{x}_1{}^j & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since the elements $\bar{x}_1{}^i \, a \, \bar{x}_1{}^j$ are linearly independent over $\mathbf{k}$ if we put

$$\bar{x}_1 a = \xi a, \qquad a\bar{x}_1 = \eta a \qquad (\xi, \eta \in \mathbf{l}),$$

then the elements

$$\xi^i \eta^i \qquad (i = 0, 1,..., j = 0, 1,...)$$

are also linearly independent over $\mathbf{k}$. So the elements

$$\begin{aligned} &\begin{pmatrix} 0 & a\bar{x}_1{}^j & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{x}_1{}^i a \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \eta^j a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi^i a \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \eta^j \xi^i a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

are also all linearly independent over $\mathbf{k}$. Thus R is infinite dimensional.

Finally observe that Q is finitely generated since W is even finitely generated as a $(\bar{T}, \bar{T})$-bimodule. But an inspection of the definitions of Q, R, and P easily yields the short exact sequence

$$0 \to R \to Q \to P \to 0.$$

Therefore Lemma 6 applies, and $m(P)$ is infinite dimensional.

4. Parafree Algebras

4.1. *Some Existence Theorems*

Our objective in this section is to prove

THEOREM 4. *Let* $\mathbf{V}$ *be a variety of associative algebras in which the free algebras are residually nilpotent and the free algebra of rank* 1 *is not nilpotent. Then for* *every positive integer q there exists a colimit of free algebras in* $\mathbf{V}$ *which is parafree of rank q but not free.*

We need some easy lemmas.

LEMMA 8. *Let* $\mathbf{V}$ *be a variety of algebras in which the free algebras are residually nilpotent. Then the finitely generated algebras in* $\mathbf{V}$ *are nilpotent if and only if the free algebras of rank* 1 *are nilpotent.*

Proof. Suppose F is a free algebra of rank 1 in $\mathbf{V}$ on x. In the event that F is not finite dimensional it is clear that $F \cong \widetilde{\mathbf{k}[x]}$, the $\mathbf{k}$-algebra of polynomials in x without constant terms. Clearly F is then not nilpotent, and so neither is any other free algebra in $\mathbf{V}$ of positive rank. If F is finite dimensional, the residual nilpotence of F ensures that $F^m = 0$ for some m. It follows that every algebra in $\mathbf{V}$ satisfies the identity $x^m = 0$. So by a theorem of Levitzki [22] every finitely generated algebra in $\mathbf{V}$ is nilpotent, as desired.

COROLLARY 8.1. *If some finitely generated algebra in* $\mathbf{V}$ *is not nilpotent, then the free algebra of rank* 1 *in* $\mathbf{V}$ *is isomorphic to* $\widetilde{\mathbf{k}[x]}$.

Corollary 8.1 is an immediate consequence of the proof of Lemma 8.

COROLLARY 8.2. *Let G be a free algebra of rank $q(< \infty)$ in* $\mathbf{V}$ *on $x_1 ,..., x_q$. If the free algebras in* $\mathbf{V}$ *are not nilpotent then the elements*

$$x_1 + x_1^2,..., x_q$$

generate a proper subalgebra H of G.

Proof. Let φ be the homomorphism of G into itself defined by

$$x_1 \mapsto x_1 , x_2 \mapsto 0,..., x_q \mapsto 0.$$

Then H maps onto the subalgebra H_1 of $F = \widetilde{\mathbf{k}[x_1]}$ generated by $x_1 + x_1^2$. Since $H_1 \neq F$, $H \neq G$ as required.

We also need the following theorem of Lewin [24] which we record as

LEMMA 9. *If X is part of a basis modulo G^2 for the residually nilpotent algebra G and if G is free in some variety* $\mathbf{V}$ *then the subalgebra generated by X is again a free algebra in* $\mathbf{V}$ *freely generated by X.*

An algebra A is termed *Hopfian* if every homomorphism of A onto itself is an automorphism. It follows readily from the fact that every finite dimensional algebra is Hopfian that

LEMMA 10. *A finitely generated residually nilpotent algebra is Hopfian.*

Finally we need

LEMMA 11. *Let* **V** *be a variety of algebras in which the free algebras are residually nilpotent. Furthermore, let* G_i *be a free algebra of finite rank* q *for* $i = 1, 2, \ldots$. *If* $G_1 \leqslant G_2 \leqslant \cdots$, *if* $G = \bigcup_{i=1}^{\infty} G_i$ *and if*

$$G_{i+1}^2 \cap G_i = G_i^2 \qquad (i = 1, 2, \ldots) \tag{3}$$

then G *is parafree in* **V** *of rank* q.

Proof. We observe that

$$G^n = \bigcup_{i=1}^{\infty} G_i^n.$$

Now it follows from (3) and Lemmas 9 and 10 that

$$G_j^n \cap G_i = G_i^n \qquad \text{whenever} \qquad j \geqslant i. \tag{4}$$

The residual nilpotence of G follows then from the residual nilpotence of the G_i.

If X is a free set of generators of G_1, then X is a basis for G modulo G^2 since each of the G_i is of rank q. It follows from (4) that

$$G/G^n \cong G_1/G_1^n \qquad (n = 1, 2, \ldots),$$

and so G is indeed parafree in **V**.

The proof of Theorem 4 is now straightforward. Let G_i be the free algebra in **V** of rank q on

$$a_{i,1}, \ldots, a_{i,q}.$$

The subalgebra G_i^* of G_{i+1} generated by

$$a_{i+1,1} + a_{i+1,1}^2, \qquad a_{i+1,2}, \ldots, a_{i+1,q}$$

is again free in **V** of rank q (Lemma 9). Moreover by Corollary 8.2, $G_i^* \neq G_{i+1}$. Identifying G_i with G_i^* under the isomorphism

$$a_{i,1} \mapsto a_{i+1,1} + a_{i+1,1}^2, \qquad a_{i,2} \mapsto a_{i+1,2}, \ldots, a_{i,q} \mapsto a_{i+1,q}$$

allows us to form the colimit (or union) G of the algebras G_i. Note that G is parafree of rank q, by Lemma 11. But G is not even finitely generated since $G^* \neq G_{i+1}$ for every i. So G is not free.

Next we prove, following Bousfield and Kan [9] (see also Baumslag and Stammbach [8])

THEOREM 5. *Let* **V** *be a variety of algebras in which the free algebras are resi-*

dually nilpotent. If F is a finitely generated free algebra in $\mathbf{V}$ *then its pronilpotent completion* $\hat{F}$ *is parafree in* $\mathbf{V}$ *of the same rank as* F.

We recall that the system $\{F/F^n \mid n = 2, 3,...\}$ of algebras, together with the homomorphisms

$$\varphi_{n,m}: F/F^n \to F/F^m \qquad (n \geqslant m),$$

defined by

$$\varphi_{n,m}: a + F^n \mapsto a + F^m \qquad (a \in F),$$

form an inverse system. By definition $\hat{F}$ is the inverse limit of this system.

Proof of Theorem 5. It is not hard to see that it suffices to prove

$$\hat{F}/\hat{F}^2 \cong F/F^2. \tag{5}$$

Suppose that F is freely generated by

$$x_1, ..., x_q.$$

Now suppose

$$\hat{f} = (f_1 + F^2, f_2 + F^3, ...) \in \hat{F}.$$

Note that

$$f_{i+1} - f_i \in F^{i+1} \qquad (i = 1, 2, ...).$$

Again it is not hard to see that the crux of the proof of (5) is to prove that whenever $f_1 \in F^2$ (and hence $f_i \in F^2$ for every i), then $\hat{f} \in \hat{F}^2$. To do so we observe that the following congruences, modulo the appropriate powers of F, hold:

$$\begin{array}{ll}
f_2 \equiv x_1 g_{11} + \cdots + x_q g_{1q} (F^3) & (g_{1i} \in F) \\
f_3 = x_1(g_{11} + g_{21}) + \cdots + x_q(g_{1q} + g_{2q}) (F^4) & (g_{2i} \in F^2) \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots & \\
f_n = x_1(g_{11} + \cdots + g_{n-1,1}) + \cdots + x_q(g_{1q} + \cdots + g_{n-1,q}) (F^{n+1}) & (g_{n-1,i} \in F^{n-1}) \\
\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots &
\end{array}$$

Put

$$\hat{x}_i = (x_i + F^2, x_i + F^3, ...) \qquad (i = 1, ..., q),$$

and

$$g_i = (g_{1i} + F^2, g_{1i} + g_{2i} + F^3, ...) \qquad (i = 1, ..., q).$$

Then the elements $\hat{x}_i$, g_i belong to $\hat{F}$ and

$$\hat{f} = \hat{x}_1 g_1 + \cdots + \hat{x}_q g_q \in \hat{F}^2.$$

COROLLARY 5.1. *If the free algebra F above is not nilpotent then $\hat{F}$ is a parafree algebra in* $\mathbf{V}$ *that is not free.*

We need only observe that F is countable whereas $\hat{F}$ is not.

4.2. *Parafree Bicommutative Algebras*

The objective of this section is to sketch the proof of

THEOREM 6. *Let* $\mathbf{M}$ *be the variety of all bicommutative algebras. Then the multiplicator of a finitely generated parafree algebra A in* $\mathbf{M}$ *of rank at least two is not finite dimensional.*

The proof of Theorem 6 follows closely that of the corresponding theorem for groups (Baumslag [7]). Indeed by appealing to Lemma 5 it suffices to prove that $A^* = A/(A')^2$ has an infinite dimensional multiplicator. To accomplish this we need to first demonstrate that A^* can be embedded in an algebra of (2×2)-matrices over a power series ring in finitely many variables, with coefficients in $\mathbf{k}$ (cf. Lewin [24], Baumslag [2]). Then by examining a related algebra of (3×3)-matrices, as in [7] (cf. the proof of Lemma 7), we put ourselves in a position to apply Lemma 6, and the result follows.

5. SOME EMBEDDING THEOREMS FOR ASSOCIATIVE ALGEBRAS

5.1. *The Proof of Theorem 7*

The objective of this section is to prove

THEOREM 7. *Let*

$$0 \to N \to A \to T \to 0$$

be a short exact sequence of associative algebras, where N is nilpotent and all of the quotients of T are finitely presented. If N^i/N^{i+1}, viewed as a (T, T)-bimodule, is finitely generated and ample for every i, then A is finitely presented.

We begin the proof of Theorem 7 by proving

LEMMA 12. *Let A be an algebra satisfying the hypothesis of Theorem 7. If $N^2 = 0$, then A and all of its quotients are finitely presented.*

Proof. Since A/N is finitely generated we can find $t_1, ..., t_q \in A$ such that

$$A/N \qquad \text{is generated by} \qquad t_1 + N, ..., t_q + N.$$

Now N is a finitely generated (T, T)-bimodule. So we can find $e_1, ..., e_r \in N$ such that N, qua ideal of A, is generated by $e_1, ..., e_r$. It follows that A is generated by

$$t_1, ..., t_q, \qquad e_1, ..., e_r. \tag{6}$$

Notice that the ampleness of the (T, T)-bimodule N ensures that there exist $v_{i,j}$, $v'_{i,j} \in A$ such that

$$t_i e_j = e_j v_{i,j}, \qquad e_j t_i = v'_{i,j} e_j \qquad (1 \leqslant i \leqslant q, 1 \leqslant j \leqslant r). \tag{7}$$

These elements $v_{i,j}$, $v'_{i,j}$ can be chosen to be words in the generators $t_1, ..., t_q$ since $N^2 = 0$.

In view of the fact that all of the quotients of T are finitely presented it follows that T is Noetherian. Hence N, viewed as a (T, T)-bimodule, is also Noetherian. Consequently N is finitely presented as a (T, T)-bimodule. The upshot of these remarks is that we can present A in the following way. First, we take (6) as a set of generators. Second, we take as defining relations the union of the sets of relations (8), (9), and (10) described below. The relations in (8) are simply the relations expressing the isomorphism $A/N \cong T$:

$$\begin{aligned} w_l(t_1, ..., t_q) = {} & w_{l,1}(t_1, ..., t_q)\, e_1 w'_{l,1}(t_1, ..., t_q) \\ & + \cdots + w_{l,r}(t_1, ..., t_q)\, e_r w'_{l,r}(t_1, ..., t_q) \qquad (l = 1, ..., m). \end{aligned} \tag{8}$$

The relations in (9) are the finitely many bimodule relations of N:

$$\begin{aligned} u_{l,1}(t_1, ..., t_q)\, e_1 u'_{l,1}(t_1, ..., t_q) + \cdots + u_{l,r}(t_1, ..., t_q)\, e_r u'_{l,r}(t_1, ..., t_q) = 0 \\ (l = 1, ..., n). \end{aligned} \tag{9}$$

The relations in (10) simply express the fact that $N^2 = 0$:

$$e_i w(t_1, ..., t_q)\, e_j = 0 \qquad (1 \leqslant i, j \leqslant r,\ w(t_1, ..., t_q) \text{ any word in } t_1, ..., t_q). \tag{10}$$

Note that the sets (8) and (9) are finite. We have to replace (10) by a finite set. To this end consider the consequences of relations (7) and the relations

$$e_i e_j = 0 \qquad (1 \leqslant i, j \leqslant r) \tag{10'}$$

If $w(t_1, ..., t_q)$ is any word in $t_1, ..., t_q$, then by invoking (7) a number of times we find

$$e_i w(t_1, ..., t_q)\, e_j = v(t_1, ..., t_q)\, e_i e_j.$$

So (7) and (10′) together imply (10). It follows that we have proved A is finitely presented.

Finally observe that if A^* is any quotient of A, then it is also the middle of a short exact sequence

$$0 \to N^* \to A^* \to T^* \to 0,$$

where every quotient of T^* is finitely presented, $(N^*)^2 = 0$, and N^* is an ample (T^*, T^*)-bimodule. Consequently A^* is finitely presented.

Theorem 7 follows immediately from Lemma 12 by induction on the class of N. Note that we have actually proved that all of the quotients of A are finitely presented.

5.2. *The Proof of Theorem* 8

It turns out to be convenient to have at hand a slight variant of Theorem 7 which we shall describe in due course. It depends on the following simple

LEMMA 13. *Suppose A and B are finitely presented algebras (over a field* $\mathbf{k}$). *Furthermore suppose both A and B have a* 1. *Then*

$$A \underset{\mathbf{k}}{\otimes} B$$

is also a finitely presented algebra with 1.

Proof. Suppose

$$A = \langle a_1, \ldots, a_m; r_1 = \cdots = r_n = 0 \rangle$$

and

$$B = \langle b_1, \ldots, b_q; s_1 = \cdots = s_q = 0 \rangle$$

are presentations of A and B, respectively. Then we claim that

$$\begin{aligned} A \underset{\mathbf{k}}{\otimes} B = \langle a_1 \otimes 1, \ldots, a_m \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_p; \\ r_1(\mathbf{a} \otimes \mathbf{1}) = \cdots = r_n(\mathbf{a} \otimes \mathbf{1}) = 0 \\ s_1(\mathbf{1} \otimes \mathbf{b}) = \cdots = s_q(\mathbf{1} \otimes \mathbf{b}) = 0 \\ [a_i \otimes 1, 1 \otimes b_j] = 0 \ (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant p) \rangle \end{aligned}$$

is a presentation of $A \otimes_{\mathbf{k}} B$, where we have denoted the word obtained from, e.g., r_1, on replacing each a_i occurring by $a_i \otimes 1$, by $r_1(\mathbf{a} \otimes \mathbf{1})$.

Consider

$$C = \langle \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_p; r_1(\boldsymbol{\alpha}) = \cdots = r_n(\boldsymbol{\alpha}) = 0,$$
$$s_1(\boldsymbol{\beta}) = \cdots = s_q(\boldsymbol{\beta}) = 0,$$
$$[\alpha_i, \beta_j] = 0 \ (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant p)\rangle.$$

The map

$$\alpha_i \mapsto a_i \otimes 1, \qquad \beta_j \mapsto 1 \otimes b_j \qquad (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant p)$$

defines a homomorphism θ' of C onto $A \otimes_{\mathrm{k}} B$. Conversely observe that the mappings $a_i \mapsto \alpha_i$ $(1 = 1,\ldots, m)$, $b_j \mapsto \beta_j$ $(j = 1,\ldots, p)$ define homomorphisms φ and μ of A and B respectively into C. Moreover the map θ: $A \times B \to C$ defined by

$$(a, b) \to a\varphi + b\mu$$

is bilinear. So θ factors through $A \otimes_{\mathrm{k}} B$ and hence determines a homomorphism, again denoted θ, from $A \otimes_{\mathrm{k}} B$ onto C. Clearly θ' and θ are inverses and this establishes the lemma.

We can now prove

THEOREM 7′. *Let T be a finitely presented algebra with* 1 *and suppose M is a finitely presented (T, T)-bimodule. Then the semidirect product A of M and $T \otimes_{\mathrm{k}} T^{\mathrm{op}}$ is a finitely presented algebra.*

Proof. M is certainly an ample $(T \otimes_{\mathrm{k}} T^{\mathrm{op}})$-bimodule. Moreover, $T \otimes_{\mathrm{k}} T^{\mathrm{op}}$ is finitely presented, by Lemma 13. Finally M is a finitely presented $(T \otimes_{\mathrm{k}} T^{\mathrm{op}}, T \otimes_{\mathrm{k}} T^{\mathrm{op}})$-bimodule, because it is already a finitely presented (T, T)-bimodule. The proof of Lemma 12 now yields the finiteness of the number of relations needed to define A.

We are now in a position to prove

THEOREM 8. *Let A be a finitely generated associative algebra with* 1 *containing an ideal N with $N^2 = 0$. If every quotient of A/N is finitely presented, then A can be embedded in a finitely presented algebra A^* which closely resembles A; specifically A^* contains an ideal N^* such that*

$$\text{(i)}\ \ N^{*2} = 0 \qquad \textit{and} \qquad \text{(ii)}\ \ A^*/N^* \cong A/N \underset{\mathrm{k}}{\otimes} A/N.$$

Proof. By a result of Lewin [24] A can be embedded in an algebra B with 1 with the following properties:

(i) B contains an ideal C such that $C^2 = 0$;

(ii) $B/C \cong A/N$;

(iii) B is finitely generated;

(iv) B splits over C, i.e., there is a subalgebra T of B such that $B = C \oplus T$.

It follows from (i)–(iv) that C is a finitely generated (T, T)-bimodule and hence finitely presented by virtue of the hypothesis on $T\,(\simeq A/N)$.

Now we may view C as a $(T \otimes_{\mathbf{k}} T^{\mathrm{op}}, T \otimes_{\mathbf{k}} T^{\mathrm{op}})$-bimodule. Thus we can form the semidirect product A^* of C with $T \otimes_{\mathbf{k}} T^{\mathrm{op}}$ (see, e.g., Cartan and Eilenberg [10, p. 293]). By Theorem 7′ then A^* is finitely presented, and by its very construction has the desired properties (with $N^* = C$).

Note now that a finitely generated Lie-nilpotent algebra T, as well as its quotients, is finitely presented (Lemma 2). So Theorem 8 yields

COROLLARY 8.1. *Let A be a finitely generated associative algebra which is the middle of a short exact sequence*

$$0 \to N \to A \to T \to 0$$

in which $N^2 = 0$ and T is Lie-nilpotent. Then A can be embedded in a finitely presented algebra A^* which is also the middle of a short exact sequence of exactly the same kind.

5.3. *The Proof of Theorems* 9 *and* 10

THEOREM 9. *Let A be a finitely generated algebra of lower triangular matrices with coefficients in a commutative algebra B. Then A can be embedded in a finitely presented algebra of lower triangular matrices with coefficients in B.*

Proof. Let n be the degree of the matrix algebra A. Now A is finitely generated. Hence we can find a finitely generated commutative $\mathbf{k}$-algebra C such that

$$A \leqslant \mathrm{Tr}(n, C) = D,$$

where $\mathrm{Tr}(n, C)$ is the algebra of *all* lower triangular matrices of degree n with coefficients in C. If N is the ideal of all those matrices in D with zero on the main diagonal we find N is nilpotent, and we have the short exact sequence

$$0 \to N \to D \to T \to 0,$$

where T is a finitely generated commutative algebra. Moreover, D is finitely generated, all of the quotients N^i/N^{i+1} of N are ample (T, T)-bimodules, and all of the quotients of T are finitely presented. Consequently Theorem 7 applies, and D is finitely presented, as desired.

We come finally to

THEOREM 10. *Let F be a finitely generated free associative algebra with* 1, *and let R be an ideal of F. If F/R is finitely presented, then F/R^n can be embedded in a finitely presented algebra A containing an ideal S such that*

$$\text{(i)} \quad S^n = 0 \qquad \textit{and} \qquad \text{(ii)} \quad A/S \cong F/R \bigotimes_{\mathrm{k}} F/R.$$

Theorem 10 is proved by induction on n. This induction rests on the following reformulation of a theorem of Lewin [24]:

LEMMA 14. *Let F be a free associative algebra with* 1 *freely generated by $\{x_i \mid i \in I\}$ and $R \trianglelefteq F$. Let M be a free $(F/R, F/R)$-bimodule freely generated by the elements $\{e_i \mid i \in I\}$, and let A be the semidirect product of M with F/R. Then the subalgebra B of A generated by the elements*

$$\{(x_i + R) + e_i \mid i \in I\}$$

is isomorphic to F/R^2 under the mapping

$$x_i + R^2 \mapsto (x_i + R) + e_i \qquad (i \in I).$$

Theorem 10 can now readily be deduced from Theorem 7′. Indeed, suppose inductively that F/R^{n-1} can be embedded in a finitely presented algebra B, say. We present B as a quotient of a finitely generated free algebra E, with 1, whose rank p is at least that of F:

$$B \cong E/Q.$$

Let M be the free $(E/Q, E/Q)$-bimodule on $e_1, \ldots, e_p$. Now form the semidirect product P of M with $(E/Q) \otimes_{\mathrm{k}} (E/Q)^{\mathrm{op}}$. Then by Theorem 7′ P is finitely presented—M is free and hence finitely presented, and $(E/Q) \otimes_{\mathrm{k}} (E/Q)$ is finitely presented by Lemma 13. Now by Lewin's theorem (Lemma 14) F/R^n is embedded in P; indeed if F is free on $x_1, \ldots, x_l$ and if $x_i + R^{n-1}$ is viewed as an element of B, then the elements

$$\{x_i + R^{n-1} + e_i \mid i = 1, \ldots, l\}$$

generate a copy of F/R^n. The conditions described in the conclusion of Theorem 10 follow by inspecting the proof above.

6. PARAFREE SEMIGROUPS ARE FREE

6.1. Suppose throughout that S is a semigroup with zero. We shall need the following:

LEMMA 15. *Suppose S is nilpotent and that X generates S modulo $\mathbf{c}_2(S)$. Then X generates S.*

Proof. Let S^2 be the subsemigroup of S consisting of those elements which can be expressed as a product of (at least) two elements of S. Now suppose $a \in S$, $a \notin S^2$. Since X generates S modulo $\mathbf{c}_2(S)$, we can find $x_1, \ldots, x_q \in X$ such that

$$(a, x_1 x_2 \cdots x_q) \in \mathbf{c}_2(S).$$

Now by the very definition of $\mathbf{c}_2(S)$ and the condition on a it follows that

$$a = x_1 x_2 \cdots x_q, \qquad q = 1,$$

i.e., $a \in X$. In other words if $a \notin S^2$, then $a \in X$.

Now suppose $a \in S^2$, $a \neq 0$. Then

$$a = b_1 b_2 \cdots b_r \qquad (r \geqslant 2, b_1, \ldots, b_r \in S),$$

where $b_i \notin S^2$. So $b_i \in X$ for each i, and hence X generates S as required.

Next we prove

LEMMA 16. *Suppose that S is residually nilpotent. If X generates S modulo $\mathbf{c}_2(S)$ then $X \cup \{0\}$ generates S.*

Proof. By Lemma 15, X generates S modulo $\mathbf{c}_n(S)$ for every n. Let T be the subsemigroup of S generated by $X \cup \{0\}$. Suppose $a \in S$, $a \notin T$. Then for each $n \geqslant 1$ there exist $t_n \in T$ such that

$$(a, t_n) \in \mathbf{c}_n(S).$$

Since $a \notin T$ it follows that for $n = 1, 2, \ldots$

$$a = a_{n,1} \cdots a_{n,n} \qquad (a_{n,i} \in S).$$

Now the residual nilpotence of S ensures that any element which can be expressed as an n-fold product for every n is the zero element of S; i.e., $a = 0$. This contradicts our assumption that $a \notin T$, which yields the conclusion $S = T$ as desired.

6.2. We are now in a position to prove Theorem 12, i.e., that a parafree semigroup S is free. To this end let

$$X = \{s \in S \mid s \notin S^2\}.$$

Then X generates S modulo $\mathbf{c}_2(S)$, and hence by Lemma 16, X generates S. Now it follows by a standard argument that any minimal set of generators of a free nilpotent semigroup freely generates that semigroup. So modulo $\mathbf{c}_n(S)$, X

freely generates $S/\mathbf{c}_n(S)$. It follows that if F is a free semigroup with zero freely generated by a set X' in a one-to-one correspondence

$$\theta\colon x' \mapsto x \qquad (x' \in X',\ x \in X),$$

with X then the homomorphism θ^* of F onto S defined by θ induces an isomorphism of $F/\mathbf{c}_n(F)$ onto $S/\mathbf{c}_n(S)$ for every n. Since F is residually nilpotent, θ^* is an isomorphism; i.e., S is free.

7. Some Embedding Theorems for Semigroups

7.1. *The Proof of Theorem* 13

We recall that a semigroup $S \in \mathscr{X}$ if (i) S is finitely generated; (ii) S is the disjoint union of its commutative subsemigroups A and T; (iii) A is an ideal of S.

Theorem 13. *If $S \in \mathscr{X}$ then S can be embedded in a finitely presented semigroup in $\mathscr{X}$.*

Proof. Consider $\mathbb{Q}S$, the semigroup algebra over the (field $\mathbb{Q}$ of rational numbers. We may assume without loss of generality that S is a semigroup with zero. Then

$$\mathbb{Q}A \trianglelefteq \mathbb{Q}S$$

and

$$\mathbb{Q}S = \mathbb{Q}A \oplus \mathbb{Q}T.$$

It follows from Corollary 4.3 of Lemma 4 that $((\mathbb{Q}S)')^2 = 0$. So by Corollary 8.1, $\mathbb{Q}S$ can be embedded in a finitely presented algebra B. B is a semidirect product of an ideal M and a finitely generated commutative subalgebra C:

$$B = M \oplus C.$$

We recall that $M^2 = 0$ and that M is an ample (C, C)-bimodule. It follows that we can find subsemigroups N and T_1 of the multiplicative semigroup comprising B, forgetting its other structure, satisfying the following conditions:

(a) S is contained in the subsemigroup S_1 generated by N and T_1;

(b) $N \subseteq M$, $T_1 \subseteq C$;

(c) there is a finite subset X of N such that N is the (semigroup) ideal of S_1 generated by X;

(d) T_1 is the subsemigroup of S_1 generated by the finite set $t_1,..., t_q$, $t'_1,..., t'_q$;

(e) if $x \in X$ then

$$t_i x = x t'_i, \qquad x t_i = t'_i x \qquad (i = 1,..., q).$$

It follows that S_1 is finitely generated. Indeed, S_1 is actually finitely presented. To see this consider $\mathbb{Q}S_1$. Then $\mathbb{Q}S_1$ is the semidirect product of its ideal $\mathbb{Q}N$ and its commutative subalgebra $\mathbb{Q}T_1$. Since $(\mathbb{Q}N)^2 = 0$ we may view $\mathbb{Q}N$ as a $(\mathbb{Q}T_1, \mathbb{Q}T_1)$-bimodule. The very definition of S_1 ensures that this bimodule is ample. Hence, by Theorem 7, $\mathbb{Q}S_1$ is finitely presented. So S_1 is finitely presented by Theorem 11. Since S_1 is clearly contained in $\mathscr{X}$, this completes the proof of Theorem 13.

7.2. *Final Comments*

It is easy enough to formulate and prove analogs for semigroups of the other theorems we have about associative algebras. For example, every finitely generated semigroup of lower triangular matrices over a commutative algebra can be embedded in a finitely presented semigroup of lower triangular matrices.

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